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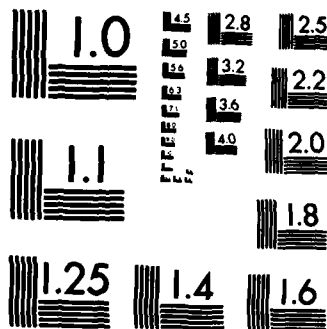
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ITEM #20, CONTINUED:

A distribution F is less dispersed than a distribution G if  $\int_{-\infty}^x F^{-1}(t) dt \leq \int_{-\infty}^x G^{-1}(t) dt$  for all  $0 < x < 1$  ( $F \leq_{disp} G$ ). We generalize a characterization of dispersive ordering of Shaked (1982) J. Appl. Prob. concerning sign changes of  $F_{sub C} - G$ , where  $F_{sub C}$  is a translate of F. We then use this generalization plus total positivity to develop a simple proof of a characterization of dispersive distributions due to Lewis and Thompson (1981) J. Appl. Prob.; a distribution H is dispersive if  $F \leq_{disp} G \Rightarrow H * F \leq_{disp} H * G$ .

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**Dispersive Ordering Results**

by

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**Key Words:** Dispersive ordering, dispersive distribution, total positivity, sign change, log concave density, variation diminishing property, partial ordering of distributions.

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## Dispersive Ordering Results

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### ABSTRACT

A distribution  $F$  is less dispersed than a distribution  $G$  if  $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$  for all  $0 < \alpha < \beta < 1$  ( $F \stackrel{\text{disp}}{\leq} G$ ). We generalize a characterization of dispersive ordering of Shaked (1982) J. Appl. Prob. concerning sign changes of  $F_c - G$ , where  $F_c$  is a translate of  $F$ . We then use this generalization plus total positivity to develop a simple proof of a characterization of dispersive distributions due to Lewis and Thompson (1981) J. Appl. Prob.; a distribution  $H$  is dispersive if  $F \stackrel{\text{disp}}{\leq} G \Rightarrow H * F \stackrel{\text{disp}}{\leq} H * G$ .

### 1. Introduction.

Dispersive ordering is a partial ordering of distributions according to their degree of dispersion. In this note we (1) generalize a characterization of dispersive ordering by Shaked (1982) and (2) using this generalization, develop a simple proof of a characterization by Lewis and Thompson (1981) of distributions which preserve dispersive ordering under convolution.

More precisely, for a distribution function (d.f.)  $F$ , define  $F^{-1}(\alpha) = \inf\{t: F(t) \geq \alpha\}$  ( $= \sup\{t: F(t) < \alpha\}$ ) for  $0 < \alpha < 1$ . Then, we have:

Definition. A d.f.  $F$  is less dispersed than a d.f.  $G$  if  $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$  for all  $0 < \alpha < \beta < 1$ ; we write  $F \stackrel{\text{disp}}{\leq} G$ .

Saunders and Moran (1978) show that the gamma distributions  $\{F_a\}$  with shape parameter  $a$  increase in dispersive ordering as  $a$  increases; a similar result holds for the ratio of gamma distributions. Saunders (1978) shows how to construct optimal detectors of bright spots in point processes using the notion of dispersive ordering. Shaked (1982) develops a number of useful characterizations of dispersive ordering in the smooth case in which  $F$  and  $G$  are absolutely continuous with interval supports. Of particular relevance to our results is his characterization in terms of sign changes of  $F_c - G$ , where  $F_c$  is a translate of  $F$ .

Lewis and Thompson (1981) present a relatively lengthy proof that a distribution  $H$  preserves  $\stackrel{\text{disp}}{<}$  under convolution if and only if  $H$  is absolutely continuous with log concave density  $h$  (Theorem 2 below).

In our note, we extend Shaked's sign change characterization to the general case (Theorem 1 below). Then, using this extension, we bring to bear the powerful tools of total positivity (see Karlin, 1968), tailor-made for the study of sign change, to give a shorter, simpler proof of the Lewis and Thompson result.

## 2. Extension of the Shaked Characterization.

Let  $S(x_1, \dots, x_n)$  denote the number of sign changes of the sequence  $x_1, \dots, x_n$ , where zero terms are discarded. Let  $S(f)$  denote the number of sign changes of the function  $f$  defined on  $(-\infty, \infty)$ ; specifically,

$$S(f) = \sup S[f(t_1), \dots, f(t_m)],$$

the supremum being taken over all  $t_1 < t_2 < \dots < t_m$ ,  $m = 2, 3, \dots$ .

Finally, let  $S_c \equiv S(F_c - G)$  for d.f.'s  $F$  and  $G$ .

Theorem 1.  $F \stackrel{\text{disp}}{\leq} G \iff$  for each real  $c$ , (a)  $S(F_c - G) \leq 1$  and  
(b) if  $S_c = 1$ , then  $F_c - G$  changes sign from  $-$  to  $+$ .

Proof.  $\Rightarrow$  For fixed  $c$ , let  $t_0$  satisfy  $G(t_0) > F_c(t_0)$ . Let  $t_* = \inf\{t: G(s) \geq F_c(s) \text{ for } t \leq s \leq t_0\}$ . We will show that  $t_* = -\infty$ . From this it will follow that  $F_c - G$  can never change from  $+$  to  $-$ . This will complete the proof.

Suppose  $t_* > -\infty$ . By the definition of  $t_*$  and the right continuity of  $F$  and  $G$ ,  $G(t_*) \geq F_c(t_*)$  and  $G(t_* - \epsilon) < F_c(t_* - \epsilon)$  for sufficiently small  $\epsilon > 0$ . Fix such an  $\epsilon$ ; then let  $\alpha = F_c(t_* - \epsilon)$  and  $\beta = G(t_0)$ . It follows that

$$G^{-1}(\alpha) \geq t_* - \epsilon \geq F_c^{-1}(\alpha)$$

and

$$G^{-1}(\beta) \leq t_0 < F_c^{-1}(\beta).$$

Thus

$$G^{-1}(\beta) - G^{-1}(\alpha) < F_c^{-1}(\beta) - F_c^{-1}(\alpha),$$

contradicting  $F \stackrel{\text{disp}}{\leq} G$ . This proves the  $\Rightarrow$  part.

$\Leftarrow$  Let  $H_n$  denote the exponential d.f. with mean  $\mu_n \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $F_n = H_n * F$  and  $G_n = H_n * G$ , where  $*$  denotes convolution. Since  $H_n$  has a log concave density and since  $S_c \leq 1$  for all real  $c$ , with a  $-$  to  $+$  sign change if  $S_c = 1$ , then by the variation diminishing theorem of Karlin (1968), Chap. 5, it follows that  $S(F_{nc} - G_n) \leq 1$ , with a  $-$  to  $+$  sign change if  $S(F_{nc} - G_n) = 1$ . Thus  $F_n \stackrel{\text{disp}}{\leq} G_n$  by Theorem 2.1 of Shaked (1981) since  $F_n$  and  $G_n$  are both absolutely continuous with interval supports.



Since  $F_n \rightarrow F$  ( $G_n \rightarrow G$ ) in distribution as  $n \rightarrow \infty$ , it follows that  $F_n^{-1} \rightarrow F^{-1}$  ( $G_n^{-1} \rightarrow G^{-1}$ ) at continuity points of  $F^{-1}$  ( $G^{-1}$ ). Thus

$$(2.1) \quad F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for  $0 < \alpha < \beta < 1$  at continuity points of both  $F$  and  $G$ . Since  $F^{-1}$  and  $G^{-1}$  are left continuous, this shows that (2.1) holds for all  $\alpha$  and  $\beta$  satisfying  $0 < \alpha < \beta < 1$ . This completes the proof of the theorem. ||

### 3. Dispersive Distributions.

Definition. A d.f.  $H$  is said to be dispersive if  $H * F \stackrel{\text{disp}}{\leq} H * G$  whenever  $F \stackrel{\text{disp}}{\leq} G$ .

Next we present a simpler proof of Theorem 8 of Lewis and Thompson (1981) characterizing dispersive d.f.'s. The simplification derives from the use of total positivity which is usually the appropriate way to treat problems of sign change.

Theorem 2. Let  $H$  be a nondegenerate d.f.. Then  $H$  is dispersive  $\iff H$  is absolutely continuous with a log concave density.

Proof.  $\Leftarrow$  Let  $F \stackrel{\text{disp}}{\leq} G$ . From Theorem 1, for each real  $c$ ,  $S_c = S(F_c - G) \leq 1$ , with a - to + sign change if  $S_c = 1$ . Since  $H$  has a log concave density, then by the variation diminishing theorem of Karlin (1968), Chap. 5,  $S[(H * F)_c - (H * G)] \leq 1$ , with a - to + sign change if  $S[(H * F)_c - (H * G)] = 1$ . It follows from Theorem 1 that  $H * F \stackrel{\text{disp}}{\leq} H * G$ .

$\Rightarrow$  We use essentially Lewis and Thompson's argument, but avoid unnecessary details of their proof. Our proof is based on their elementary

proof of the "only if" part of their Theorem 7; we restate this for completeness.

Lemma. Let  $F$  be dispersive and twice continuously differentiable. Then  $F$  has a log concave density.

To complete the proof of Theorem 2, let  $H$  be dispersive and let  $\phi_\sigma$  denote a normal distribution with mean 0 and variance  $\sigma^2$ . From the first part of Theorem 2, note that  $\phi_\sigma$  is dispersive, and thus,  $\phi_\sigma * H$  is also dispersive. Since  $\phi_\sigma * H$  is infinitely differentiable, it follows from the Lemma above that  $\phi_\sigma * H$  has a log concave density. Since  $\phi_\sigma * H \rightarrow H$  in distribution as  $\sigma \rightarrow 0$ ,  $H$  has a log concave density. (See Ibragimov, 1956; note that a distribution is "strongly unimodal" in Ibragimov's terminology iff its density is log concave.)

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